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Monotonic analysis over ordered topological vector spaces: IV

A. R. Doagooei · Hossein Mohebi

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Abstract In this paper, we present an extension for non-negative increasing and co-radiant (ICR) functions over a topological vector space. We characterize the essential results of abstract convexity such as support set, subdifferential and polarity of these functions. We also give some characterizations of a certain kind of polarity and separation property for non-convex radiant and co-radiant sets.

Keywords Monotonic analysis · ICR function · Radiant set · Co-radiant set · Abstract convexity

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1 Introduction

It is well-known that every proper and lower semi-continuous convex function can be expressed as a pointwise supremum of a family of affine functions majorized by it (see [10]). It is natural to see what happens if we replace affine functions by a certain class of functions which are so-called elementary functions. This gave rise to the subject of Abstract Convexity (for more details see [9,11,12]). It is well-known that some classes of increasing functions are abstract convex, for example, the class of Increasing and Positively Homogeneous (IPH) functions and the class of Increasing and Convex-Along-Rays (ICAR) functions. The first studies of these functions were carried out over the cones in topological vector spaces (see [3,4]). Some suitable extensions for these functions defined over the whole of topological vector spaces were obtained in [2,7,8].

A. R. Doagooei · H. Mohebi (✉)
Department of Mathematics, Mahani Mathematical Research Center,
Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: hmohebi@mail.uk.ac.ir; ho_mohebi@yahoo.com

A. K. Doagooei
e-mail: doagooei@graduate.uk.ac.ir

Increasing and Co-radiant (ICR) functions are another class of increasing functions which are abstract convex. The theory of ICR functions can be applied in mathematical economics (see, e.g., [6]), where quasi-concave ICR functions have been studied. The first characterization of these functions has been shown in [10] over the cone R^n_+. This was generalized in [5], where ICR functions defined over cones in a vector space.

In this paper, we generalize ICR functions defined on a topological vector space and give some characterizations of these functions. As an application, we present a kind of separation property for radiant and co-radiant sets.

The layout of the paper is as follows. In Sect. 2, we collect definitions, notations and preliminary results related to ICR functions. In Sects. 3 and 4, we obtain some results of abstract convexity for ICR functions and characterize their subdifferential and support sets. We study polarity of ICR functions in Sect. 5. Finally, the relation between IPH and ICR functions will be given in Sect. 6.

2 Preliminaries

Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.

A function $f : X \longrightarrow [0, +\infty]$ is called co-radiant if $f(\lambda x) \geq \lambda f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is easy to see that f is co-radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in X$ and all $\lambda \geq 1$. The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$.

Definition 2.1 An increasing function $f : X \to R$ is called concave-along-rays (ICAR), if for each $x \in X$ the function $f_x(\alpha) = f(\alpha x)$ is concave for all $\alpha \in (0, +\infty)$.

The support set of a $\triangle$-convex function is defined by:

$$supp(f, \triangle) := \{l \in \triangle \ : \ l(x) \leq f(x) \ \forall \ x \in X\},$$

where $\triangle$ is the set of elementary functions. Also, the $\triangle$-subdifferential at a point $x_0 \in X$ is defined by:

$$\partial_{\triangle} f(x_0) := \{l \in \triangle \ : \ f(x) - f(x_0) \geq l(x) - l(x_0) \ \forall \ x \in X\}.$$

The following definitions are well-known.

(i) A non-empty subset A of X is called downward, if $x \in A$, $x' \in X$ and $x' \leq x$ imply $x' \in A$.
(ii) A non-empty subset B of X is called upward, if $x \in B$, $x' \in X$ and $x \leq x'$ imply $x' \in B$.
(iii) A non-empty subset A of X is radiant, if $x \in A$ and $0 < \lambda < 1$ imply $\lambda x \in A$. Also, a subset B of X is co-radiant, if $x \in B$ and $\lambda > 1$ imply $\lambda x \in B$.

Now, we present some examples of ICR functions.

Example 2.1 It is easy to check that an ICAR function f such that $f(0) \geq 0$ is ICR. In fact, we have for each $x \in X$ and $\lambda \in (0, 1]$:

$$f(\lambda x) = f_x(\lambda) = f_x(\lambda + (1 - \lambda)0) \geq \lambda f_x(1) + (1 - \lambda) f(0) \geq \lambda f(x).$$

Example 2.2 An increasing positively homogeneous function f of degree δ, where $0 < \delta \leq 1$, is ICR.

3 Abstract convexity of non-negative ICR functions

Some definitions related to the abstract convexity have been introduced in [11]. In this section, we discuss on abstract convexity with respect to a certain class of ICR functions. We also investigate subdifferential and a special kind of polarity of ICR functions.

Consider the function $l : X \times X \times R_{++} \longrightarrow [0, +\infty]$ defined by:

$$l(x, y, \alpha) := \max\{0 \le \lambda \le \alpha : \lambda y \le x\}, \tag{3.1}$$

(we use the convention $\max \emptyset = 0$).

In the following, we give some properties of this function.

Proposition 3.1 *For every $x,\ y,\ x',\ y' \in X;\ \gamma \in (0, 1];\ \mu,\ \alpha,\ \alpha' \in R_{++}$, one has*

$$l(\mu x, y, \alpha) = \mu l\left(x, y, \frac{\alpha}{\mu}\right), \tag{3.2}$$

$$l(x, \mu y, \alpha) = \frac{1}{\mu} l(x, y, \mu\alpha), \tag{3.3}$$

$$x \le x' \implies l(x, y, \alpha) \le l(x', y, \alpha), \tag{3.4}$$

$$y \le y' \implies l(x, y', \alpha) \ge l(x, y, \alpha), \tag{3.5}$$

$$\alpha \le \alpha' \implies l(x, y, \alpha) \le l(x, y, \alpha'), \tag{3.6}$$

$$l(\gamma x, y, \alpha) \ge \gamma l(x, y, \alpha), \tag{3.7}$$

$$l(x, \gamma y, \alpha) \le \frac{1}{\gamma} l(x, y, \alpha), \tag{3.8}$$

$$l(x, y, \alpha) = \alpha \quad \Leftrightarrow \quad \alpha y \le x. \tag{3.9}$$

Proof We only prove parts (3.2) and (3.7). For (3.2) we have:

$$\begin{aligned} l(\mu x, y, \alpha) &= \max\{0 \le \lambda \le \alpha : \lambda y \le \mu x\} \\ &= \max\{0 \le \lambda \le \alpha : \frac{\lambda}{\mu} y \le x\} \\ &= \max\{0 \le \mu\tilde{\lambda} \le \alpha : \tilde{\lambda} y \le x\} \\ &= \mu l\left(x, y, \frac{\alpha}{\mu}\right). \end{aligned}$$

Finally, (3.7) follows from (3.2) and (3.6). □

Example 3.1 Let $X = \mathbb{R}^n$ and S be the cone $\mathbb{R}^n_+$ of all vectors in $\mathbb{R}^n$ with non-negative coordinates. Let $I = \{1, 2, ..., n\}$. Each vector $x \in \mathbb{R}^n$ generates the following sets of indices:

$$I_+(x) = \{i \in I : x_i > 0\}, \quad I_0(x) = \{i \in I : x_i = 0\}, \quad I_-(x) = \{i \in I : x_i < 0\}.$$

Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Denote by $\frac{c}{x}$ the vector with coordinates

$$\left(\frac{c}{x}\right)_i := \begin{cases} \frac{c}{x_i}, & i \notin I_0(x), \\ 0, & i \in I_0(x). \end{cases}$$

Then, for each $x,\ y \in \mathbb{R}^n$, we have

$$l(x, y, \alpha) = \begin{cases} \min\left\{\min_{i \in I_+(y)} \frac{x_i}{y_i}\ ,\ \alpha\right\}, & x \in K_y^+, \\ 0, & x \notin K_y^+, \end{cases}$$

where

$$K_y^+ := \left\{x \in \mathbb{R}^n : \forall\, i \in I_+(y) \cup I_0(y),\ \ x_i \geq 0;\ \max_{i \in I_-(y)} \frac{x_i}{y_i} \leq \min_{i \in I_+(y)} \frac{x_i}{y_i}\right\}.$$

We can also introduce the function $u : X \times X \times R_{++} \longrightarrow [0, +\infty]$ defined by

$$u(x, y, \beta) := min\{\lambda \geq \beta : \beta y \geq x\}, \tag{3.10}$$

(with the convention $min\emptyset = +\infty$).

The following proposition gives us some properties of the function u.

Proposition 3.2 *For every $x,\ y,\ x',\ y' \in X;\ \gamma \in (0, 1];\ \mu,\ \beta,\ \beta' \in R_{++}$, one has*

$$u(\mu x, y, \beta) = \mu u\left(x, y, \frac{\beta}{\mu}\right), \tag{3.11}$$

$$u(x, \mu y, \beta) = \frac{1}{\mu} u(x, y, \mu\beta), \tag{3.12}$$

$$x \leq x' \implies u(x, y, \beta) \leq u(x', y, \beta), \tag{3.13}$$

$$y \leq y' \implies u(x, y', \beta) \geq u(x, y, \beta), \tag{3.14}$$

$$\beta \leq \beta' \implies u(x, y, \beta) \leq u(x, y, \beta'), \tag{3.15}$$

$$u(\gamma x, y, \beta) \geq \gamma u(x, y, \beta), \tag{3.16}$$

$$u(x, \gamma y, \beta) \leq \frac{1}{\gamma} u(x, y, \beta), \tag{3.17}$$

$$u(x, y, \beta) = \beta \iff \beta y \geq x. \tag{3.18}$$

In the following proposition we give the relation between the functions l and u.

Proposition 3.3 *Let l and u be as the above. Then, for all $x,\ y \in X$ and all $\mu > 0$, we have:*

$$l(x, y, \mu) u\left(y, x, \frac{1}{\mu}\right) = 1, \tag{3.19}$$

(with the convention $0 \times (+\infty) = (+\infty) \times 0 = 1$).

Proof This is an immediate consequence of the definition l and u. □

Theorem 3.1 *Let $f : X \to [0, +\infty]$ be a function. Then the following assertions are equivalent:*

(i) f is ICR.
(ii) $\lambda f(y) \leq f(x)$ for all $x,\ y \in X$ and all $\lambda \in (0, 1]$ such that $\lambda y \leq x$.
(iii) $l(x, y, \alpha) f(\alpha y) \leq \alpha f(x)$ for all $x,\ y \in X$ and all $\alpha \in R_{++}$, with the convention $0 \times (+\infty) = 0$.
(iv) $u(x, y, \beta) f(\beta y) \geq \beta f(x)$ for all $x,\ y \in X$ and all $\beta \in R_{++}$, with the convention $0 \times (+\infty) = +\infty$.

Proof

(i)⇒ (ii). It is obvious.

(ii)⇒ (iii). Let $\alpha > 0$ and $x,\ y \in X$ be arbitrary. If $l(x, y, \alpha) \neq 0$, then $l(x, y, \alpha) y \leq x$, and also $0 < \frac{l(x,y,\alpha)}{\alpha} \leq 1$. Thus, by hypothesis and the fact that $\frac{l(x,y,\alpha)}{\alpha}(\alpha y) \leq x$, we conclude that $\frac{l(x,y,\alpha)}{\alpha} f(\alpha y) \leq f(x)$. Therefore, holds. Trivially, (iii) holds if $l(x, y, \alpha) = 0$.

(iii)⇒ (i). Now, let $y \leq x$. Then, by (3.9), $l(x, y, 1) = 1$, and (iii) implies that $f(y) = f(y)l(x, y, 1) \leq f(x)$. So, f is increasing. Moreover, according to (3.7), we have $\lambda l(x, x, 1) \leq l(\lambda x, x, 1)$ for all $\lambda \in (0, 1]$ and all $x \in X$, Thus

$$\lambda f(x) = \lambda l(x, x, 1) f(x) \leq l(\lambda x, x, 1) f(x) \leq f(\lambda x).$$

Hence, f is ICR.

(i)⇒ (iv). Let $u(x, y, \beta) \neq +\infty$. Then $\frac{u(x,y,\beta)}{\beta} \geq 1$, and also $\frac{u(x,y,\beta)}{\beta}(\beta y) \geq x$. Since f is ICR, it follows that

$$\frac{u(x, y, \beta)}{\beta} f(\beta y) \geq f\left(\frac{u(x, y, \beta)}{\beta}(\beta y)\right) \geq f(x).$$

Also, (iv) holds if $u(x, y, \beta) = +\infty$ because $0 \times (+\infty) = +\infty$.

Finally, the proof of the implication $(iv) \Rightarrow (i)$ can be done in a similar manner as the proof of the implication $(iii) \Rightarrow (i)$. □

Remark 3.1 Consider $\frac{a}{b} = 0$, then we should consistently have $\frac{b}{a} = (\frac{a}{b})^{-1} = 0^{-1} = +\infty$, even though $\frac{a}{b}$ and $\frac{b}{a}$ yield the same expression $\frac{0}{0}$, when both a and b are equal 0. In Theorem 3.1 (iii) we use the convention $0 \times 0^{-1} = 0$, whereas in Theorem 3.1 (iv) we take $0 \times 0^{-1} = +\infty$. In fact, this second choice is necessary for the sake of consistency with the first one.

We could solve this apparent inconsistency regarding notation by introducing two different operations, a "lower" division and an "upper" division, similarly to the lower addition and upper addition often used in Abstract Convex Analysis, but we preferred to avoid this in order to keep our notation as simple as possible.

Now, we are going to show that each non-negative ICR function is supremally generated by a certain class of ICR functions.

Assume that $y \in X$ and $\alpha \in R_{++}$. Consider the function $l_{(y,\alpha)} : X \to [0, +\infty]$ defined by $l_{(y,\alpha)}(x) = l(x, y, \alpha)$. Also, let $L := \{l_{(y,\alpha)} : y \in X,\ \alpha \in R_{++}\}$ be the set of elementary functions.

Remark 3.2 By (3.4) and (3.7), the function $l_{(y,\alpha)}$ is an ICR function.

Theorem 3.2 *Let $f : X \to [0, +\infty]$ be a function. Then f is ICR if and only if there exists a set $A \subset L$ such that*

$$f(x) = \sup_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x).$$

In this case, one can take $A = \{l_{(y,\alpha)} \in L : f(\alpha y) \geq \alpha\}$. Hence, f is ICR if and only if f is L-convex.

Proof We only prove that each ICR function $f : X \to [0, +\infty]$ satisfies

$$f(x) = \sup_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x).$$

According to Theorem 3.1, we have $l_{(y,\alpha)}(x) f(\alpha y) \leq \alpha f(x)$ for all $x,\ y \in X$ and all $\alpha \in R_{++}$. So, if $x \in X$ and $l_{(y,\alpha)} \in A$ are arbitrary, then $l_{(y,\alpha)}(x) \leq f(x)$. Let $0 < f(x) < +\infty$. Then $l_{(\frac{x}{f(x)}, f(x))} \in A$, and since $l_{(\frac{x}{f(x)}, f(x))}(x) = f(x)$, it follows that $f(x) = \max_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x)$.

Now, consider $f(x) = 0$. Let $l_{(y,\alpha)} \in A$ be such that $l_{(y,\alpha)}(x) \neq 0$. According to Theorem 3.1, we have $l_{(y,\alpha)}(x)f(\alpha y) \leq \alpha f(x) = 0$, which implies that $f(\alpha y) = 0$. But, $0 = f(\alpha y) \geq \alpha$, and this is a contradiction. So, $l_{(y,\alpha)} = 0$ for all $l_{(y,\alpha)} \in A$.

Finally, if $f(x) = +\infty$ and $\alpha > 1$, then $f(\frac{x}{\alpha}) \geq \frac{1}{\alpha} f(x) = +\infty$. Put, $y_\alpha = \frac{x}{\alpha}$. Trivially, we get $f(y_\alpha) = +\infty \geq \alpha$ for all $\alpha > 1$, and thus $l_{(y,\alpha)} \in A$. Therefore, we have

$$f(x) = +\infty = \sup_{\alpha} l_{(y_\alpha,\alpha)}(x) \leq \sup_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x) \leq f(x).$$

Hence, the proof is complete. □

As the above, we can also show that each ICR function $f : X \to [0, +\infty]$ is infimally generated by a certain class of ICR functions. Let $y \in X$ and $\beta \in R_{++}$. Consider the function $u_{(y,\beta)} : X \to [0, +\infty]$ defined by $u_{(y,\beta)}(x) = u(x, y, \beta)$. Also, let $U := \{u_{(y,\beta)} : y \in X,\ \beta \in R_{++}\}$ be the set of elementary functions.

Remark 3.3 By (3.13) and (3.16), the function $u_{(y,\beta)}$ is an ICR function.

The proof of the following theorem is similar to the one of Theorem 3.2, and therefore we omit it.

Theorem 3.3 *Let $f : X \to [0, +\infty]$ be a function. Then f is ICR if and only if there exists a set $B \subset U$ such that*

$$f(x) = \inf_{u_{(y,\alpha)} \in B} u_{(y,\alpha)}(x).$$

In this case, one can take $B = \{u_{(y,\beta)} \in U : f(\beta y) \leq \beta\}$. Hence, f is ICR if and only if f is U-concave.

Recall that a function f is inf-abstract-convex if $f(x) = \inf_\alpha f_\alpha(x)$ such that each f_α is abstract-convex.

Corollary 3.1 *If $f : X \to [0, +\infty]$ is ICR, then f is inf-abstract-convex.*

Proof It follows from Theorem 3.2 that $f(x) = \inf_{u_{(y,\alpha)}} u_{(y,\alpha)}(x)$. Since each $u_{(y,\beta)}$ is ICR, it follows from Theorem 3.2 that $u_{(y,\beta)} = \sup_{l_{(y,\alpha)}} l_{(y,\alpha)}(x)$, and the proof is complete. □

4 Subdifferential and support sets

In this section, we present a description of support set and the L-subddiferential of an ICR function f defined on a topological vector space X, and we investigate some properties of support sets in X.

Recall that the support set of a $\triangle$-convex function is defined by:

$$supp(f, \triangle) := \{l \in \triangle : l(x) \leq f(x)\ \forall\, x \in X\},$$

where $\triangle$ is the set of elementary functions.

Proposition 4.1 *Let $f : X \to [0, +\infty]$ be an ICR function. Then*

$$supp(f, L) = \{l_{(y,\alpha)} \in L : f(\alpha y) \geq \alpha\}.$$

Proof Let $l_{(y,\alpha)} \in supp(f, L)$. We have $l_{(y,\alpha)}(x) \le f(x)$ for all $x \in X$, and so, for $x = \alpha y$, we obtain $\alpha = l_{(y,\alpha)}(\alpha y) \le f(\alpha y)$. Now, suppose that $(y, \alpha) \in X \times R_{++}$ be such that $f(\alpha y) \ge \alpha$. According to Theorem 3.1(iii), we have

$$l_{(y,\alpha)}(x) \le f(x),$$

for all $x \in X$, which completes the proof. □

Recall that for a $\triangle$-convex function $f : X \longrightarrow [0, +\infty]$, the $\triangle$-subdifferential at a point $x_0 \in X$ is defined as follows:

$$\partial_\triangle f(x_0) := \{l \in \triangle \ : \ f(x) - f(x_0) \ge l(x) - l(x_0) \ \forall \ x \in X\},$$

where $\triangle$ is the set of finite elementary functions.

Proposition 4.2 *Let $f : X \to [0, +\infty]$ be an ICR function and $x_0 \in X$ be such that $f(x_0) \ne 0, +\infty$. Then*

$$\{l_{(y,\alpha)} \ : \ f(\alpha y) \ge \alpha, \ l_{(y,\alpha)}(x_0) = f(x_0)\} \subset \partial_L f(x_0).$$

Moreover, $\partial_L f(x_0) \ne \emptyset$.

Proof Let $l_{(y,\alpha)} \in \{l_{(y,\alpha)} \ : \ f(\alpha y) \ge \alpha, \ l_{(y,\alpha)}(x_0) = f(x_0)\}$. By Proposition 4.1, we have $f(\alpha y) \ge \alpha$ if and only if $l_{(y,\alpha)}(x) \le f(x)$ for all $x \in X$. This, together with the fact that $f(x_0) = l_{(y,\alpha)}(x_0)$ imply that $l_{(y,\alpha)} \in \partial_l f(x_0)$. Now, put $y = \frac{x_0}{f(x_0)}$ and $\alpha = f(x_0)$, which implies that $f(\alpha y) = \alpha$ and $l_{(y,\alpha)}(x_0) = f(x_0)$. Hence, $l_{(y,\alpha)} \in \partial_L f(x_0)$. □

Theorem 4.1 *Let $f : X \to [0, +\infty]$ be an ICR function and $x_0 \in X$ be such that $f(x_0) \ne +\infty$. Then*

$$\{l_{(y,\alpha)} \ : \ f(x_0) \le l_{(y,\alpha)}(x_0), \ \alpha - l_{(y,\alpha)}(x_0) \le f(\alpha y) - f(x_0)\} \subset \partial_L f(x_0).$$

Moreover, the equality holds if and only if $\inf_{x \in X} f(x) = 0$.

Proof Let $D = \{l_{(y,\alpha)} \ : \ f(x_0) \le l_{(y,\alpha)}(x_0), \ \alpha - l_{(y,\alpha)}(x_0) \le f(\alpha y) - f(x_0)\}$ and $l_{(y,\alpha)} \in D$ be arbitrary. Since $\frac{l_{(y,\alpha)}(x)}{\alpha} \le 1$ and $0 \le l_{(y,\alpha)}(x_0) - f(x_0)$, it follows that

$$\frac{l_{(y,\alpha)}(x)}{\alpha}(\alpha - f(\alpha y)) \le \frac{l_{(y,\alpha)}(x)}{\alpha}(l_{(y,\alpha)}(x_0) - f(x_0)) \le l_{(y,\alpha)}(x_0) - f(x_0), \quad (4.1)$$

for all $x \in X$. According to Theorem 3.1(iii), we have $\frac{l_{(y,\alpha)}(x)}{\alpha} f(\alpha y) \le f(x)$ for all $x \in X$. This, together with (4.1) imply that

$$l_{(y,\alpha)}(x) - f(x) \le \frac{l_{(y,\alpha)}(x)}{\alpha}(\alpha - f(\alpha y)) \le l_{(y,\alpha)}(x_0) - f(x_0),$$

for all $x \in X$. Hence, $l_{(y,\alpha)} \in \partial_L f(x_0)$.

Now,assume that $\inf_{x \in X} f(x) = 0$ and $l_{(y,\alpha)} \in \partial_L f(x_0)$. By definition we have

$$l_{(y,\alpha)}(x) - l_{(y,\alpha)}(x_0) \le f(x) - f(x_0), \quad (4.2)$$

for all $x \in X$. This means that $-l_{(y,\alpha)}(x_0) \le l_{(y,\alpha)}(x) - l_{(y,\alpha)}(x_0) \le f(x) - f(x_0)$. Thus, $f(x_0) - l_{(y,\alpha)}(x_0) \le \inf_{x \in X} f(x) = 0$, which implies that $f(x_0) \le l_{(y,\alpha)}(x_0)$. Moreover, put $x = \alpha y$ in (4.2), we obtain

$$\alpha - l_{(y,\alpha)}(x_0) \le f(\alpha y) - f(x_0). \quad (4.3)$$

Hence, $l_{(y,\alpha)} \in D$.

Now, we are going to show that if $D = \partial_L f(x_0)$, then $\inf_{x\in X} f(x) = 0$. Let $\alpha > f(0) - \inf_{x\in X} f(x)$ be arbitrary, we claim that $l_{(0,\alpha)} \in \partial_L f(0)$. For this end, we have

$$l_{(0,\alpha)}(x) = \begin{cases} \alpha, & x \in S \\ 0, & x \notin S. \end{cases} \tag{4.4}$$

Let $x \in S$. Since $f(0) \leq f(x)$, then, by (4.4), we have

$$l_{(0,\alpha)}(x) - l_{(0,\alpha)}(0) = 0 \leq f(x) - f(0) \ \ \forall\, x \in S.$$

Also, since $\alpha > f(0) - \inf_{x\in X} f(x)$, then, by (4.4), we have

$$l_{(0,\alpha)}(x) - l_{(0,\alpha)}(0) = 0 - \alpha \leq f(x) - f(0) \ \ \forall\, x \in X \setminus S.$$

So, $l_{(0,\alpha)} \in \partial_L f(0)$ for all $\alpha > f(0) - \inf_{x\in X} f(x)$. Moreover, since $D = \partial_L f(0)$, we conclude that

$$f(0) \leq l_{(0,\alpha)}(0) \leq \alpha \quad \forall\, \alpha > f(0) - \inf_{x\in X} f(x).$$

As $\alpha \to f(0) - \inf_{x\in X} f(x)$, we get $f(0) \leq f(0) - \inf_{x\in X} f(x)$, and this implies that $\inf_{x\in X} f(x) = 0$. □

Corollary 4.1 *Let $f : X \to [0, +\infty]$ be an ICR function. Define the function $g : X \to [0, +\infty]$ by $g(x) := f(x) - \inf_{x\in X} f(x)$. Assume that g is an ICR function. Then we have*

$$\partial_L f(x_0) = \{l_{(y,\alpha)} \ : \ f(x_0) \leq l_{(y,\alpha)}(x_0) + \inf_{x\in X} f(x), \ \alpha - l_{(y,\alpha)}(x_0) \leq f(\alpha y) - f(x_0)\}.$$

Proof Since $\partial_L f(x_0) = \partial_L g(x_0)$ and $\inf_{x\in X} g(x) = 0$, the result follows from Theorem 4.1. □

In the rest of this section, we introduce $X \times R_{++}$-support sets for an ICR function which are essential to characterize polar sets.

Let $f : X \to [0, +\infty]$ be a function. The lower $(X \times R_{++})$-support set of f, $supp_l(f, X \times R_{++})$, is defined by:

$$supp_l(f, X \times R_{++}) := \{(y, \alpha) \in X \times R_{++} \ : \ l_{(\frac{y}{\alpha},\alpha)} \leq f\}. \tag{4.5}$$

Also, we define the upper $(X \times R_{++})$-support set of f, $supp_u(f, X \times R_{++})$, by:

$$supp_u(f, X \times R_{++}) := \{(y, \alpha) \in X \times R_{++} \ : \ u_{\left(\frac{y}{\beta},\beta\right)} \geq f\}. \tag{4.6}$$

Let $W \subset X \times R_{++}$, recall that the α-section of W (W^α) is defined by $W^\alpha := \{y \in X \ : (y, \alpha) \in W\}$. Also, y-section of W (W_y) is defined by $W_y := \{\alpha \in R_{++} \ : \ (y, \alpha) \in W\}$.

Remark 4.1 Let $f : X \to [0, +\infty]$ be a function. According to (3.5), (3.6) and (4.5), $supp_l(f, X \times R_{++})$ is a radiant set and has the upward α-section, also the y-section of $supp_l(f, X \times R_{++})$ is a normal and closed set in R_{++}.

Remark 4.2 Let $f : X \to [0, +\infty]$ be a function. According to (3.14), (3.15) and (4.6), $supp_u(f, X \times R_{++})$ is a co-radiant set and has the downward α-section, also the y-section of $supp_u(f, X \times R_{++})$ is a co-normal and closed set in R_{++}.

Proposition 4.3 *Let $W \subset X \times R_{++}$ and $W \neq \emptyset$. Then the following assertions are equivalent:*

(i) *W is radiant, the section W^α is upward for all $\alpha > 0$, and for all $y \in X$ the section W_y is normal and closed in R_{++}.*
(ii) *There is a unique ICR function $f : X \to [0, +\infty]$ such that $supp_l(f, X \times R_{++}) = W$.*
(iii) *There is a unique function $f : X \to [0, +\infty]$ such that $supp_l(f, X \times R_{++}) = W$.*

Proof

(i)⇒ (ii). Define the function f by $f(y) := sup\{\alpha > 0 : (y, \alpha) \in W\}$ for all $y \in X$ (with the convention $sup\ \emptyset = 0$). We are going to show that f is an ICR function. For this end, let $y_1 \le y_2$ and $\alpha \in W_{y_1}$. Then $y_1 \in W^\alpha$. Since W^α is upward, then we obtain $y_2 \in W^\alpha$, which implies that $\alpha \in W_{y_2}$. On the other hand, we have $W_{y_1} \subset W_{y_2}$. Thus, $f(y_1) \le f(y_2)$. Hence, f is increasing.
Now, assume that $0 < \lambda \le 1$ and $y \in X$ be arbitrary. Then, we get:

$$\begin{aligned} f(\lambda y) &= \sup\{\alpha : (\lambda y, \alpha) \in W\} \\ &\ge sup\left\{\alpha : \left(y, \frac{\alpha}{\lambda}\right) \in W\right\} \\ &= sup\{\lambda\beta : (y, \beta) \in W\} \\ &= \lambda f(y). \end{aligned}$$

Therefore, f is a co-radiant function. Now, we are going to show that $W = supp_l(f, X \times R_{++})$. For this end, let $(y, \alpha) \in W$, then $f(y) \ge \alpha$. Since f is an ICR function, it follows from Proposition 4.1 that $l_{(\frac{y}{\alpha}, \alpha)} \le f$, which implies that $(y, \alpha) \in supp_l(f, X \times R_{++})$, and so $W \subset supp_l(f, X \times R_{++})$.
For the converse inclusion, let $(y, \alpha) \in supp_l(f, X \times R_{++})$. This implies that $f(y) \ge \alpha$. Since W_y is closed and normal in R_{++}, we conclude that $\alpha \in W_y$, and hence $(y, \alpha) \in W$. Therefore, $W = supp_l(f, X \times R_{++})$.
Moreover, f is unique. Indeed, suppose that there exists an ICR function $h : X \to [0, +\infty]$ such that $W = supp_l(h, X \times R_{++})$. Let $x \in X$ be such that $h(x) \ne 0$. Then, by Proposition 4.1, we have $l_{(y,\alpha)} \le h$ if and only if $h(\alpha y) \ge \alpha$. So, we deduce that $l_{\left(\frac{x}{h(x)}, h(x)\right)} \in supp_l(h, X \times R_{++}) = W = supp_l(f, X \times R_{++})$, which means that $l_{\left(\frac{x}{h(x)}, h(x)\right)} \le f$. Then, by (3.9) and the fact that f is positive, we obtain $h(x) \le f(x)$ for all $x \in X$. By a similar argument we can get $f(x) \le h(x)$ for all $x \in X$.

(ii)⇒ (iii). It is obvious.

(iii)⇒ (i). It is an immediate consequence of Remark 4.1.

□

The proof of the following proposition is similar to the one of Proposition 4.3, and therefore we omit it.

Proposition 4.4 *Let $Q \subset X \times R_{++}$ and $Q \ne \emptyset$. Then the following assertions are equivalent:*

(i) *Q is co-radiant, the section Q^β is downward for all $\beta > 0$, and for all $x \in X$ the section Q_x is co-normal and closed in R_{++}.*
(ii) *There is a unique ICR function $f : X \to [0, +\infty]$ such that $supp_u(f, X \times R_{++}) = Q$.*
(iii) *There is a unique function $f : X \to [0, +\infty]$ such that $supp_u(f, X \times R_{++}) = Q$.*
Furthermore, the function f of (ii) is defined by $f(x) := \inf\{\beta : (x, \beta) \in Q\}$ for all $x \in X$ (with the convention $\inf \emptyset = 0$).

5 Polarity of ICR functions and co-radiant sets

In the this section, we introduce the polarity of ICR functions and some co-radiant sets. Also, we present a separation theorem for these sets.

Definition 5.1 The lower polar function of $f : X \to [0, +\infty]$ is the function $f_l^0 : L \to [0, +\infty]$ defined by

$$f_l^0\left(l_{(y,\alpha)}\right) := \sup_{x\in X} \frac{l_{(y,\alpha)}(x)}{f(x)} \quad \forall\, l_{(y,\alpha)} \in L, \tag{5.1}$$

(with the convention $\frac{0}{0} = 0$).

Proposition 5.1 *Let $f : X \to [0, +\infty]$ be a function. Then*

$$f_l^0\left(l_{(y,\alpha)}\right) \geq \frac{\alpha}{f(\alpha y)} \quad \forall\, l_{(y,\alpha)} \in L.$$

Moreover, f is an ICR function if and only if

$$f_l^0\left(l_{(y,\alpha)}\right) = \frac{\alpha}{f(\alpha y)} \quad \forall\, l_{(y,\alpha)} \in L. \tag{5.2}$$

Proof By (5.1), it follows that $f_l^0(l_{(y,\alpha)}) \geq \frac{l_{(y,\alpha)}(x)}{f(x)}$ for all $x \in X$. This implies that $\frac{\alpha}{f(\alpha y)} = \frac{l_{(y,\alpha)}(\alpha y)}{f(\alpha y)} \leq f_l^0(l_{(y,\alpha)})$. Now, let f be an ICR function and $x,\ y \in X,\ \alpha > 0$ be arbitrary. According to Theorem 3.1(iii), we have $l_{(y,\alpha)}(x) f(\alpha y) \leq \alpha f(x)$. This, together with the convention $\frac{0}{0} = 0$ imply that

$$f_l^0\left(l_{(y,\alpha)}\right) \leq \frac{\alpha}{f(\alpha y)} \quad \forall\, l_{(y,\alpha)} \in L.$$

Hence, we get (5.2). The rest of the proof follows from (5.1) and Proposition 3.3. □

Corollary 5.1 *Let $f : X \to [0, +\infty]$ be an ICR function. Then*

$$supp(f, L) = \{l_{y,\alpha)} \ :\ f_l^0\left(l_{(y,\alpha)}\right) \leq 1\ \}.$$

We can also define the upper polar functions which are defined by the elementary functions $u_{(y,\beta)}$.

Definition 5.2 The upper polar function of $f : X \to [0, +\infty]$ is the function $f_u^0 : U \to [0, +\infty]$ defined by

$$f_u^0\left(u_{(y,\beta)}\right) := \inf_{x\in X} \frac{u_{(y,\beta)}(x)}{f(x)} \quad \forall\, u_{(y,\beta)} \in U,$$

(with the convention $\frac{0}{0} = +\infty$).

Proposition 5.2 *Let $f : X \to [0, +\infty]$ be a function. Then*

$$f_u^0\left(u_{(y,\beta)}\right) \leq \frac{\beta}{f(\beta y)} \quad \forall\, u_{(y,\beta)} \in U.$$

Moreover, f is an ICR function if and only if

$$f_u^0\left(u_{(y,\beta)}\right) = \frac{\beta}{f(\beta y)} \quad \forall\, u_{(y,\beta)} \in u.$$

Definition 5.3 Let $W \subset X \times R_{++}$. The left polar set of W (W^l) is defined by:

$$W^l := \left\{(x,\beta) \in X \times R_{++} \ : \ l_{(\frac{y}{\alpha},\alpha)}(x) \le \beta, \ \forall\,(y,\alpha) \in W\right\}. \tag{5.3}$$

Proposition 5.3 *Let $W \subset X \times R_{++}$. Then*

$$W^l = supp_u(h_W, X \times R_{++}),$$

where the function $h_W : X \to [0,+\infty]$ is defined by:

$$h_W(y) := sup\{\alpha > 0 \ : \ (y,\alpha) \in W\}, \quad \forall\, y \in X, \tag{5.4}$$

(with the convention $\sup \emptyset = 0$*).*

Proof By (5.3) and (3.19), we conclude that

$$\begin{aligned} W^l &= \left\{(x,\beta) \in X \times R_{++} \ : \ l_{(\frac{y}{\alpha},\alpha)}(x) \le \beta, \ \forall\ (y,\alpha) \in W\right\} \\ &= \left\{(x,\beta) \in X \times R_{++} \ : \ \beta u_{\left(x,\frac{1}{\alpha}\right)}\left(\frac{y}{\alpha}\right) \ge 1, \ \forall\ (y,\alpha) \in W\right\} \\ &= \left\{(x,\beta) \in X \times R_{++} \ : \ u_{\left(\frac{x}{\beta},\beta\right)}(y) \ge \alpha, \ \forall\ (y,\alpha) \in W\right\} \\ &= \left\{(x,\beta) \in X \times R_{++} \ : \ u_{\left(\frac{x}{\beta},\beta\right)}(y) \ge h(y), \ \forall\ y \in X\right\} \\ &= supp_u(h_W, X \times R_{++}). \end{aligned}$$

□

Definition 5.4 Let $W \subset X \times R_{++}$. The right polar set of W (W^r) is defined by:

$$W^r := \{(y,\alpha) \in X \times R_{++} \ : \ l_{(\frac{y}{\alpha},\alpha)}(x) \le \beta, \ \forall\,(x,\beta) \in W\}. \tag{5.5}$$

Similar to the Proposition 5.3, we have the following result.

Proposition 5.4 *Let $W \subset X \times R_{++}$. Then*

$$W^r = supp_l(e_W, X \times R_{++}),$$

where the function $e_W : X \to [0,+\infty]$ is defined by:

$$e_W(x) := \inf\{\beta > 0 \ : \ (x,\beta) \in W\}, \quad \forall\, x \in X, \tag{5.6}$$

(with the convention $\inf \emptyset = +\infty$*).*

Remark 5.1 Let $W \subset X \times R_{++}$ and $W \neq \emptyset$. According to (3.5), (3.6) and (5.3), we have W^r is a radiant set, the section $(W^r)^\alpha$ is upward for all $\alpha > 0$, and for all $y \in X$ the section $(W^r)_y$ is a normal and closed set in R_{++}.

Also, by (3.4), (3.7) and (5.5), we have W^l is a co-radiant set, the section $(W^l)^\beta$ is downward for all $\beta > 0$, and for all $x \in X$ the section $(W^l)_x$ is a co-normal and closed set in R_{++}.

The sets which are closed under the closure operators $W \to W^{rl}$ and $W \to W^{lr}$ are identified in the following theorem.

Theorem 5.1 *Let $W \subset X \times R_{++}$. Then the following assertions are true:*

(i) *One has* $W = W^{rl}$ *if and only if W is co-radiant and has the downward* β*-section and closed co-normal* x*-section for all* $\beta > 0$ *and all* $x \in X$.
(ii) *One has* $W = W^{lr}$ *if and only if W is radiant and has the upward* α*-section and closed normal* y*-section for all* $\alpha > 0$ *and all* $y \in X$.

Proof We only prove the part (i). Let $W = W^{rl}$. By Remark 5.1, we have W is co-radiant and has the downward β-section and closed co-normal x-section.

Conversely, let W be co-radiant and has the downward β-section and closed co-normal x-section. Then, by Proposition 4.4, there exists a unique ICR function f such that $W = supp_u(f, X \times R_{++})$. In view of Proposition 4.4 and (5.6), we conclude that $f = e_W$. Moreover, Proposition 5.4 and the fact that $f = e_W$ imply that $W^r = supp_l(f, X \times R_{++})$. Also, according to Remark 5.1, we have W^r is radiant and has the upward α-section and closed normal y-section. Thus, by Proposition 4.3 there exists a unique function g such that $supp_l(g, X \times R_{++}) = W^r$. By (5.4) and the definition of g, we obtain $g = h_{W^r}$. Since g is unique and $supp_l(g, X \times R_{++}) = W^r = supp_l(f, X \times R_{++})$, then $f = h_{W^r}$. Now, by Proposition 5.3, we have:

$$W^{rl} = supp_u(h_{W^r}, X \times R_{++}) = supp_u(f, X \times R_{++}) = W,$$

which completes the proof. □

Many applications of convexity are based on the separation property. Some notions of separability of radiant and co-radiant sets has been introduced and studied in [13]. In the following theorem, we give a kind of separation property for a certain class of co-radiant sets by an elementary ICR function.

Theorem 5.2 *Let* $W \subset X \times R_{++}$. *Then the following assertions are equivalent:*

(i) W *is a co-radiant set and has the downward* β*-section and closed co-normal* x*-section for all* $\beta > 0$ *and all* $x \in X$.
(ii) *For each* $(x_0, \beta_0) \notin W$, *there exists* $(y, \alpha) \in X \times R_{++}$ *such that*

$$\frac{1}{\beta} l_{(y,\alpha)}(x) \leq 1 < \frac{1}{\beta_0} l_{(y,\alpha)}(x_0) \ \ \forall\ (x, \beta) \in W. \tag{5.7}$$

Proof

(i)⇒ (ii). Let $(x_0, \beta_0) \notin W$. It follows from Theorem 5.1 that $(x_0, \beta_0) \notin W^{rl}$. This, together with the definition of W^r imply that there exists $(\tilde{y}, \alpha) \in W^r$ such that $l_{(\frac{\tilde{y}}{\alpha},\alpha)}(x_0) > \beta_0$ and $l_{(\frac{\tilde{y}}{\alpha},\alpha)}(x) \leq \beta$ for all $(x, \beta) \in W$. Let $y = \frac{\tilde{y}}{\alpha}$. Thus, $l_{(y,\alpha)}$ satisfies (5.7).

(ii)⇒ (i). According to Theorem 5.1, we only show that $W^{rl} \subset W$. For this end, assume that $(x_0, \beta_0) \in W^{rl}$ and $(x_0, \beta_0) \notin W$, so by hypothesis there exists $(y, \alpha) \in X \times R_{++}$ such that

$$\frac{1}{\beta} l_{(y,\alpha)}(x) \leq 1 < \frac{1}{\beta_0} l_{(y,\alpha)}(x_0) \ \ \forall\ (x, \beta) \in W. \tag{5.8}$$

The left inequality in (5.8) shows that $(\alpha y, \alpha) \in W^r$. Then, from $(x_0, \beta_0) \in W^{rl}$ and $(\alpha y, \alpha) \in W^r$ we conclude that $l_{(y,\alpha)}(x_0) \leq \beta_0$, and this contradicts the right inequality in (5.8). □

In the following, we present a kind of separation property for a certain class of radiant sets by an elementary ICR function.

Theorem 5.3 *Let $W \subset X \times R_{++}$. Then the following assertions are equivalent:*

(i) W a is radiant set and has the upward α-section and closed normal y-section for all $\alpha > 0$ and all $y \in X$.
(ii) For each $(y_0, \alpha_0) \notin W$, there exists $(x, \beta) \in X \times R_{++}$ such that

$$\frac{1}{\alpha_0} u_{(x,\beta)}(y_0) < 1 \leq \frac{1}{\alpha} u_{(x,\beta)}(y) \ \ \forall \ (y, \alpha) \in W.$$

6 ICR functions and IPH functions

Abstract convexity of IPH functions on topological vector spaces has been studied in [2] and [8]. It is well-known that IPH and ICR functions are closely related (see, for example, [10]). In this section, we characterize subdifferential of ICR functions by means of IPH functions which are simpler. For this end, we need the following definition: Let $f : X \to [0, +\infty]$ be a function. The positively homogeneous extension function $\hat{f}$ of $f : X \times R_{++} \cup \{(0, 0)\} \to [0, +\infty]$ is defined by:

$$\hat{f}(x, \lambda) := \lambda f\left(\frac{x}{\lambda}\right), \ (x \in X, \ \lambda > 0), \ \hat{f}(0, 0) = 0.$$

We consider the natural order relation with respect to $S \times R_{++}$ on the space $X \times R_{++}$ by:

$$(x_1, c_1) \leq (x_2, c_2) \Leftrightarrow x_2 - x_1 \in S, \ c_1 \leq c_2.$$

The following result on the cone R^n_+ has been proved in [1], and can easily be extended to topological vector spaces with the same proof.

Theorem 6.1 *A function $f : X \to [0, +\infty]$ is ICR if and only if its positively homogeneous extension $\hat{f}(x, \lambda)$ is increasing.*

Let f be an ICR function. Consider its positively homogeneous extension $\hat{f}$ defined on $X \times R_{++}$. It follows from Theorem 6.1 that $\hat{f}$ is an IPH function.

The following results play a main role to reach our purpose.

Theorem 6.2 ([8], Theorem 3.2) Let $p : X \longrightarrow [0, +\infty]$ be a function. Then p is IPH if and only if p is Ω-convex, where $\Omega := \{l_y \ : \ y \in X\}$ and $l_y(x) = \max\{0 \leq \lambda \ : \ \lambda y \leq x\}$.

Theorem 6.3 ([2], Theorem 2.7) Let $p : X \to [0, \infty]$ be an IPH function, and $p(x) \neq 0, +\infty$. Then

$$\partial_\Omega p(x) = \{l_y \in \Omega \ \ : \ \ l_y(x) = p(x), \ \ p(y) = 1\}.$$

Let us now define $\tilde{L} := \{\tilde{l}_{(y,\alpha)} \ \ : \ l_{(y,\alpha)} \in L\}$, where $\tilde{l}_{(y,\alpha)}(x, c) := l_{(y, \frac{c}{\alpha})}(x), \ \forall \ (x, c) \in X \times R_{++}$.

Remark 6.1 Let $f : X \to [0, +\infty]$ be an ICR function. Then $\hat{f}$ is a $\tilde{L}$-convex function. In this case, Ω in Theorem 6.2 is exactly $\tilde{L}$. On the other hand, we have:

$$\hat{f}\,(x, c) = cf\left(\frac{x}{c}\right) = \sup_L c\, l_{(y,\alpha)}\left(\frac{x}{c}\right) = \sup_{\tilde{L}} \tilde{l}_{\left(y, \frac{1}{\alpha}\right)}\,(x, c)\,.$$

for all $x \in X$ and all $c > 0$.

Now, we give a description of subdifferential $\partial_{\tilde{L}} f(y)$.

Theorem 6.4 *Let $f : X \to [0, +\infty]$ be an ICR function, and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Then*

$$\partial_{\tilde{L}} f(x_0) = \left\{ \tilde{l}_{(y,\alpha)} \ : \ f(x_0) = l_{(y,\alpha)}(x_0), \ f(\alpha y) = \alpha \right\}.$$

Proof According to Remark 6.1, Theorem 6.1 and Theorem 6.3, we have

$$\partial_{\tilde{L}} \hat{f}(x_0, 1) = \left\{ \tilde{l}_{(y,\alpha)} \ : \ \hat{f}(x_0, 1) = \tilde{l}_{\left(y, \frac{1}{\alpha}\right)}(x_0, 1), \ \hat{f}\left(y, \frac{1}{\alpha}\right) = 1 \right\}.$$

Now, the result follows by definition of the positively homogeneous extension function $\hat{f}$. □

Example 6.1 Let $X = \mathbb{R}^n$ and S be the cone $\mathbb{R}^n_+$ of all vectors in $\mathbb{R}^n$ with non-negative coordinates. According to Example 3.1, we have

$$l(x, y, \alpha) = \begin{cases} \min\left\{\min_{i \in I_+(y)} \frac{x_i}{y_i}\ ,\ \alpha\right\}, & x \in K_y^+, \\ 0, & x \notin K_y^+, \end{cases}$$

for each $x, \ y \in \mathbb{R}^n$, where

$$K_y^+ := \left\{ x \in \mathbb{R}^n \ : \ \forall\, i \in I_+(y) \cup I_0(y), \ \ x_i \geq 0; \ \max_{i \in I_-(y)} \frac{x_i}{y_i} \leq \min_{i \in I_+(y)} \frac{x_i}{y_i} \right\}.$$

Now, assume that $f : R^n_+ \to [0, +\infty]$ be an ICR function, and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$, then

$$\partial_{\tilde{L}} f(x_0) = \left\{ \tilde{l}_{(y,\alpha)} \ : \ f(x_0) = \min\left\{ \min_{i \in I_+(y)} \frac{(x_0)_i}{y_i}\ ,\ \alpha \right\}, \ f(\alpha y) = \alpha \right\},$$

where we have:

$$\tilde{l}_{(y,\alpha)}(x, c) = \begin{cases} \min\{\min_{i \in I_+(y)} \frac{x_i}{y_i}\ ,\ \frac{c}{\alpha}\}, & x \in K_y^+, \\ 0, & x \notin K_y^+. \end{cases}$$

for all $x \in R^n_+$ and all $c > 0$.

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